

THE ℓ^1 -INDEX OF TSIRELSON TYPE SPACES

DENNY H. LEUNG AND WEE-KEE TANG

ABSTRACT. If α and β are countable ordinals such that $\beta \neq 0$, denote by $\tilde{T}_{\alpha,\beta}$ the completion of c_{00} with respect to the implicitly defined norm

$$\|x\| = \max\{\|x\|_{c_0}, \frac{1}{2} \sup \sum_{i=1}^j \|E_i x\|\},$$

where the supremum is taken over all finite subsets E_1, \dots, E_j of \mathbb{N} such that $E_1 < \dots < E_j$ and $\{\min E_1, \dots, \min E_j\} \in \mathcal{S}_\beta$. It is shown that the Bourgain ℓ^1 -index of $\tilde{T}_{\alpha,\beta}$ is $\omega^{\alpha+\beta \cdot \omega}$. In particular, if $\omega_1 > \alpha = \omega^{\alpha_1} \cdot m_1 + \dots + \omega^{\alpha_n} \cdot m_n$ in Cantor normal form and α_n is not a limit ordinal, then there exists a Banach space whose ℓ^1 -index is ω^α .

Let E be a separable Banach space not containing a copy of ℓ^1 . The complexity of the $\ell^1(n)$'s inside E may be measured by Bourgain's ℓ^1 -index [3] or by locating so called ℓ^1_α -spreading models [8]. It is easy to see that the existence of ℓ^1_α -spreading models implies a large ℓ^1 -index. In general, the implication is not reversible [7, Remark 6.6(i)]. However, suppose that T is the standard Tsirelson space constructed by Figiel and Johnson [5] (the dual of the original Tsirelson space [10]). It is known that there is a constant K such that every normalized block basic sequence in T is K -equivalent to a subsequence of the unit vector basis of T (see e.g., [4]). Using this observation, one can show that the existence of ℓ^1 -block trees in T with large indices leads to the existence of large ℓ^1_α -spreading models. The result can be used to calculate the ℓ^1 -index of T . In this paper, we show that a similar method can be applied to certain general Tsirelson type spaces. In particular, it is shown that if $\omega_1 > \alpha = \omega^{\alpha_1} \cdot m_1 + \dots + \omega^{\alpha_n} \cdot m_n$ in Cantor normal form and α_n is not a limit ordinal, then there exists a Banach space whose ℓ^1 -index is ω^α . This gives a partial answer to Question 1 in [7].

If M is an infinite subset of \mathbb{N} , denote the set of all finite, respectively infinite subsets of M by $[M]^{<\omega}$, respectively $[M]$. A subset \mathcal{F} of $[\mathbb{N}]^{<\omega}$ is *hereditary* if $G \in \mathcal{F}$ whenever $G \subseteq F \in \mathcal{F}$. \mathcal{F} is *spreading* if whenever $F = \{n_1, \dots, n_k\} \in \mathcal{F}$ with $n_1 < \dots < n_k$ and $m_1 < \dots < m_k$ satisfies $m_i \geq n_i$ for $1 \leq i \leq k$ then $\{m_1, \dots, m_k\} \in \mathcal{F}$. \mathcal{F} is *compact* if it is compact in the product topology in $2^{\mathbb{N}}$. A set \mathcal{F} of finite subsets of \mathbb{N} is called *regular* if it has all three properties. If E and F are finite subsets of \mathbb{N} , we write $E < F$, respectively $E \leq F$, to mean $\max E < \min F$, respectively $\max E \leq \min F$ ($\max \emptyset = 0$ and $\min \emptyset = \infty$). We abbreviate $\{n\} < E$ and $\{n\} \leq E$ to $n < E$ and $n \leq E$ respectively. Given $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$, a sequence of finite subsets $\{E_1, \dots, E_n\}$ of \mathbb{N} is said to be \mathcal{F} -admissible if $E_1 < \dots < E_n$ and $\{\min E_1, \dots, \min E_n\} \in \mathcal{F}$. If \mathcal{M} and \mathcal{N} are regular subsets of $[\mathbb{N}]^{<\omega}$, we let

$$\mathcal{M}[\mathcal{N}] = \{\cup_{i=1}^k F_i : F_i \in \mathcal{N} \text{ for all } i \text{ and } \{F_1, \dots, F_k\} \mathcal{M}\text{-admissible}\}$$

and

$$(\mathcal{M}, \mathcal{N}) = \{M \cup N : M < N, M \in \mathcal{M} \text{ and } N \in \mathcal{N}\}.$$

We also write \mathcal{M}^2 for $(\mathcal{M}, \mathcal{M})$. Of primary importance are the Schreier classes as defined in [1]. Let $\mathcal{S}_0 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ and $\mathcal{S}_1 = \{F \subseteq \mathbb{N} : |F| \leq \min F\}$. Here $|F|$ denotes the cardinality of F . The higher Schreier classes are defined inductively as follows. $\mathcal{S}_{\alpha+1} = \mathcal{S}_1[\mathcal{S}_\alpha]$ for all $\alpha < \omega_1$. If α is a countable limit ordinal, choose a sequence (α_n) strictly increasing to α and set

$$\mathcal{S}_\alpha = \{F : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \leq |F|\}.$$

It is clear that \mathcal{S}_α is a regular family for all $\alpha < \omega_1$. If $M = (m_1, m_2, \dots)$ is a subsequence of \mathbb{N} , let $\mathcal{S}_\alpha(M) = \{\{m_i : i \in F\} : F \in \mathcal{S}_\alpha\}$. Since \mathcal{S}_α is spreading, $\mathcal{S}_\alpha(M) \subseteq \mathcal{S}_\alpha$.

Let c_{00} be the space of all finitely supported sequences. If $F \in [\mathbb{N}]^{<\infty}$ and $a = (a_n) \in c_{00}$, let $Fa = (b_n) \in c_{00}$, where $b_n = a_n$ if $n \in F$ and 0 otherwise; also set $\sigma_F((a_n)) = \sum_{n \in F} |a_n|$. Finally, if $\mathcal{S}_0 \subseteq \mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$, define an associated norm $\|\cdot\|_{\mathcal{F}}$ on c_{00} by $\|(a_n)\|_{\mathcal{F}} = \sup_{F \in \mathcal{F}} \sigma_F((a_n))$.

Definition 1. Let α, β be countable ordinals such that $\beta \neq 0$. Define $\|\cdot\|_n$ and $\|\cdot\|'_n$, $n \in \mathbb{N}$, inductively as follows. Let $\|\cdot\|_0 = \|\cdot\|'_0 = \|\cdot\|_{\mathcal{S}_\alpha}$. If $x \in c_{00}$, set

$$\|x\|_{n+1} = \max \left\{ \|x\|_n, \sup \left\{ \frac{1}{2} \sum_{i=1}^j \|E_i x\|_n : \{E_1, \dots, E_j\} \text{ } \mathcal{S}_\beta\text{-admissible} \right\} \right\},$$

and

$$\|x\|'_{n+1} = \max \left\{ \|x\|'_n, \sup \left\{ \frac{1}{2} \sum_{i=1}^j \|E_i x\|'_n : \{E_1, \dots, E_j\} \text{ } (\mathcal{S}_\beta)^2\text{-admissible} \right\} \right\}.$$

Note that $(\|x\|_n)_{n \in \mathbb{N}}$ and $(\|x\|'_n)_{n \in \mathbb{N}}$ are increasing sequences majorized by the ℓ^1 -norm of x . Let $\|x\|_{\tilde{T}} = \lim_{n \rightarrow \infty} \|x\|_n$ and $\|x\|_{\tilde{T}}' = \lim_{n \rightarrow \infty} \|x\|'_n$. Denote by $\tilde{T}_{\alpha, \beta}$ and $\tilde{T}_{\alpha, \beta}'$ respectively the completion of c_{00} under the norms $\|\cdot\|_{\tilde{T}}$ and $\|\cdot\|_{\tilde{T}}'$. Clearly, $\tilde{T}_{0,1}$ is the Tsirelson space constructed by Figiel and Johnson [5] and $\tilde{T}_{0, \beta}$ is the space denoted by $T(\mathcal{S}_\beta, \frac{1}{2})$ in [7]. The ℓ^1 -index of $\tilde{T}_{0, \beta}$ is shown to be $\omega^{\beta \cdot \omega}$ in [7]. Here, we use a different argument to compute the ℓ^1 -indices of the spaces $\tilde{T}_{\alpha, \beta}$. The next proposition can be verified immediately.

Proposition 2. The norms $\|\cdot\|_{\tilde{T}}$ and $\|\cdot\|_{\tilde{T}}'$ satisfy the implicit equations

$$\|x\|_{\tilde{T}} = \max \left\{ \|x\|_{\mathcal{S}_\alpha}, \sup \left\{ \frac{1}{2} \sum_{i=1}^j \|E_i x\|_{\tilde{T}} : \{E_1, \dots, E_j\} \text{ } \mathcal{S}_\beta\text{-admissible} \right\} \right\}$$

and

$$\|x\|_{\tilde{T}}' = \max \left\{ \|x\|_{\mathcal{S}_\alpha}, \sup \left\{ \frac{1}{2} \sum_{i=1}^j \|E_i x\|_{\tilde{T}}' : \{E_1, \dots, E_j\} \text{ } (\mathcal{S}_\beta)^2\text{-admissible} \right\} \right\}$$

for all $x \in c_{00}$.

Proposition 4 is a close relative of Lemma 5 in [4]. It is the key observation that allows us to reduce ℓ^1 -block trees on $\tilde{T}_{\alpha,\beta}$ to subsequences of the unit vector basis (e_n) of $\tilde{T}_{\alpha,\beta}$. The following lemma is easily established by induction.

Lemma 3. *Suppose that $n_1 \leq I_1 < n_2 \leq I_2 < \dots < n_k \leq I_k$ and $|I_j| \leq 2$ for $1 \leq j \leq k$. If $\{n_1, n_2, \dots, n_k\} \in \mathcal{S}_\beta$ for some $\beta < \omega_1$, then $\bigcup_{j=1}^k I_j \in (\mathcal{S}_\beta)^2$.*

Obviously, the coordinate unit vectors (u_n) is a normalized 1-unconditional basis of $\tilde{T}_{\alpha,\beta}$. The *support* of an element $x = \sum a_n u_n \in \tilde{T}_{\alpha,\beta}$ is the set $\text{supp } x = \{n : a_n \neq 0\}$.

Proposition 4. *For every $\|\cdot\|_{\tilde{T}}$ -normalized block basis (x_1, x_2, \dots, x_p) in $\tilde{T}_{\alpha,\beta}$, and all $(a_k) \in c_{00}$,*

$$\left\| \sum_{k=1}^p a_k x_k \right\|_{\tilde{T}} \leq 2 \left\| \sum_{k=1}^p a_k e_{i_k} \right\|_{\tilde{T}}$$

for all $(a_k) \in c_{00}$, where $i_k = \max \text{supp } x_k$, and (e_n) is the unit vector basis of $\tilde{T}_{\alpha,\beta}$.

Proof. With the notation as above, we prove by induction that $\|\sum_{k=1}^p a_k x_k\|_n \leq 2 \|\sum_{k=1}^p a_k e_{i_k}\|'_n$ for all $n \in \mathbb{N} \cup \{0\}$, $(a_k) \in c_{00}$.

To establish the inequality for the case $n = 0$, let $I \in \mathcal{S}_\alpha$. Define $J = \{k : I \cap \text{supp } x_k \neq \emptyset\}$. Then

$$\begin{aligned} \sigma_I \left(\sum_{k=1}^p a_k x_k \right) &= \sum_{k=1}^p |a_k| \sigma_I(x_k) \\ &\leq \sum_{k \in J} |a_k| \|x_k\|_0 \\ &\leq \sum_{k \in J} |a_k| = \sigma_L \left(\sum_{k=1}^p a_k e_{i_k} \right), \text{ where } L = \{i_k : k \in J\}, \\ &\leq \left\| \sum_{k=1}^p a_k e_{i_k} \right\|'_0, \text{ since } L \in \mathcal{S}_\alpha. \end{aligned}$$

Suppose the proposition holds for some n . Let $\{E_1, \dots, E_q\}$ be \mathcal{S}_β -admissible. Without loss of generality, we may assume that E_1, \dots, E_q are successive integer intervals, that for all j , $E_j \cap \text{supp } x_k \neq \emptyset$ for some k , and that $i_p \leq \max E_q$. Also let I_k be the integer interval $[i_{k-1} + 1, i_k]$ ($i_0 \equiv 0$). Let $A = \{j : E_j \not\subseteq I_k \text{ for any } k\}$ and $B = \{j : j \notin A\}$. For $j \in A$, set $H_j = \{k : I_k \subseteq E_j\}$ and $G_j = \{i_k : k \in H_j\}$. Then define $F_j = (E_j \cap \{i_1, \dots, i_p\}) \setminus G_j$. Note that $F_j < G_j$ for all $j \in A$. If $j \in B$, set $G_j = E_j \cap \{i_1, \dots, i_p\}$.

It follows from Lemma 3 that $(F_j)_{j \in A} \cup (G_j)_{j=1}^q$ is $(\mathcal{S}_\beta)^2$ -admissible. Finally, let $J = \{k : k \notin \cup_{j \in A} H_j, I_k \cap (\cup E_j) \neq \emptyset\}$. Now

$$\begin{aligned}
\sum_{j=1}^q \left\| E_j \left(\sum_{k=1}^p a_k x_k \right) \right\|_n &= \sum_{j=1}^q \left\| E_j \left(\sum_{j' \in A} \sum_{k \in H_{j'}} a_k x_k + \sum_{k \in J} a_k x_k \right) \right\|_n \\
&\leq \sum_{j=1}^q \left(\left\| E_j \left(\sum_{j' \in A} \sum_{k \in H_{j'}} a_k x_k \right) \right\|_n + \left\| E_j \left(\sum_{k \in J} a_k x_k \right) \right\|_n \right) \\
&= \sum_{j \in A} \left\| E_j \left(\sum_{k \in H_j} a_k x_k \right) \right\|_n + \sum_{j=1}^q \left\| E_j \left(\sum_{k \in J} a_k x_k \right) \right\|_n \\
&\leq \sum_{j \in A} \left\| \sum_{k \in H_j} a_k x_k \right\|_n + \sum_{k \in J} |a_k| \sum_{j=1}^q \|E_j x_k\|_n \\
&\leq \sum_{j \in A} \left\| \sum_{k \in H_j} a_k x_k \right\|_n + 2 \sum_{k \in J} |a_k| \|x_k\|_{n+1} \\
&\leq 2 \left(\sum_{j \in A} \left\| \sum_{k \in H_j} a_k e_{i_k} \right\|_n' + \sum_{k \in J} |a_k| \right) \text{ by inductive hypothesis,} \\
&= 2 \left(\sum_{j \in A} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{k \in J} |a_k| \right).
\end{aligned}$$

If $k \in J$, then either $\{i_k\} = F_j$ for some $j \in A$ or $\{i_k\} = G_j$ for some $j \in B$. Therefore

$$\sum_{k \in J} |a_k| \leq \sum_{j \in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{j \in B} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n'.$$

Hence

$$\begin{aligned}
\sum_{j=1}^q \left\| E_j \left(\sum_{k=1}^p a_k x_k \right) \right\|_n &\leq 2 \sum_{j \in A} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + 2 \sum_{j \in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \\
&\quad + 2 \sum_{j \in B} \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \\
&= 2 \left(\sum_{j \in A} \left\| F_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' + \sum_{j=1}^q \left\| G_j \left(\sum_{k=1}^p a_k e_{i_k} \right) \right\|_n' \right) \\
&\leq 4 \left\| \sum_{k=1}^p a_k e_{i_k} \right\|_{n+1}', \text{ as } (F_j)_{j \in A} \cup (G_j)_{j=1}^q \text{ is } (\mathcal{S}_\beta)^2\text{-admissible.}
\end{aligned}$$

Thus

$$\frac{1}{2} \sum_{j=1}^q \left\| E_j \left(\sum_{k=1}^p a_k x_k \right) \right\|_n \leq 2 \left\| \sum_{k=1}^p a_k e_{i_k} \right\|'_{n+1}$$

whenever $\{E_1, \dots, E_q\}$ is \mathcal{S}_β -admissible. It follows that

$$\left\| \sum_{k=1}^p a_k x_k \right\|_{n+1} \leq 2 \left\| \sum_{k=1}^p a_k e_{i_k} \right\|'_{n+1}.$$

This completes the induction. \square

Let α, β be countable ordinals. Define the families (\mathcal{F}_n) , (\mathcal{F}'_n) , (\mathcal{G}_n) and (\mathcal{G}'_n) inductively as follows: $\mathcal{F}_0 = \mathcal{F}'_0 = \mathcal{S}_\alpha$, $\mathcal{G}_1 = \mathcal{S}_\beta$, $\mathcal{G}'_1 = (\mathcal{S}_\beta)^2$, for all $n \in \mathbb{N}$,

$$\mathcal{F}_{n+1} = \mathcal{S}_\beta[\mathcal{F}_n], \mathcal{F}'_{n+1} = (\mathcal{S}_\beta)^2[\mathcal{F}'_n], \mathcal{G}_{n+1} = \mathcal{S}_\beta[\mathcal{G}_n] \text{ and } \mathcal{G}'_{n+1} = (\mathcal{S}_\beta)^2[\mathcal{G}'_n].$$

It is easily verified that $\mathcal{G}_n[\mathcal{S}_\alpha] = \mathcal{F}_n$, $\mathcal{G}'_n[\mathcal{S}_\alpha] = \mathcal{F}'_n$, $\mathcal{G}_n[\mathcal{S}_\beta] = \mathcal{G}_{n+1}$ and $\mathcal{G}'_n[(\mathcal{S}_\beta)^2] = \mathcal{G}'_{n+1}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, denote the norms $\|\cdot\|_{\mathcal{F}_n}$ and $\|\cdot\|_{\mathcal{F}'_n}$ by ρ_n and ρ'_n respectively.

Proposition 5. *For all $a \in c_{00}$, and all $n \in \mathbb{N} \cup \{0\}$, $\|a\|_{\tilde{T}} \geq \frac{1}{2^n} \rho_n(a)$.*

Proof. The proof is by induction on n . The case $n = 0$ is clearly true by definition. Suppose the result holds for some n . Let $E \in \mathcal{F}_{n+1}$. Then $E = \bigcup_{i=1}^j E_i$, where $E_1, \dots, E_j \in \mathcal{F}_n$, $E_1 < \dots < E_j$, and $\{E_1, \dots, E_j\}$ is \mathcal{S}_β -admissible. For any $a = (a_k) \in c_{00}$,

$$\sum_{k \in E} |a_k| = \sum_{i=1}^j \sum_{k \in E_i} |a_k| \leq \sum_{i=1}^j \rho_n(E_i a) \leq 2^n \sum_{i=1}^j \|E_i a\|_{\tilde{T}} \leq 2^{n+1} \|a\|_{\tilde{T}}.$$

Since $E \in \mathcal{F}_{n+1}$ is arbitrary, the result follows. \square

Proposition 6. *For all $a \in c_{00}$, and all $n \in \mathbb{N} \cup \{0\}$,*

$$\|a\|_{\tilde{T}} \leq \sum_{i=0}^n \frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+1}} \sup \left\{ \sum_{i=1}^j \|E_i a\|_{\tilde{T}} : \{E_1, \dots, E_j\} \text{ } \mathcal{G}'_{n+1}\text{-admissible} \right\}.$$

Proof. The proof is by induction on n . The case $n = 0$ holds by Proposition 4. Assume the result holds for some n . Let $a \in c_{00}$. Suppose $\{E_1, \dots, E_j\}$ is \mathcal{G}'_{n+1} -admissible. Let $I = \{i : \|E_i a\|_{\tilde{T}} = \rho'_0(E_i a)\}$ and $J = \{1, 2, \dots, j\} \setminus I$. For each $i \in I$, choose $D_i \subseteq E_i$, $D_i \in \mathcal{S}_\alpha$, such that $\rho'_0(E_i a) = \sum_{k \in D_i} |a_k|$. Now $D = \bigcup_{i \in I} D_i \in \mathcal{G}'_{n+1}[\mathcal{S}_\alpha] = \mathcal{F}'_{n+1}$. Hence

$$(1) \quad \sum_{i \in I} \|E_i a\|_{\tilde{T}} = \sum_{k \in D} |a_k| \leq \rho'_{n+1}(a).$$

On the other hand, for each $i \in J$, there exist $(\mathcal{S}_\beta)^2$ -admissible sets $\{E_1^i, \dots, E_{k_i}^i\}$, $E_1^i \cup \dots \cup E_{k_i}^i \subseteq E_i$ such that

$$\|E_i a\|_{\tilde{T}} = \frac{1}{2} \sum_{p=1}^{k_i} \|E_p^i a\|_{\tilde{T}}.$$

Now $\{\min E_p^i : i \in J, 1 \leq p \leq k_i\} \in \mathcal{G}'_{n+1} \left[(\mathcal{S}_\beta)^2 \right] = \mathcal{G}'_{n+2}$. Hence $(E_p^i)_{i \in J, 1 \leq p \leq k_i}$ is \mathcal{G}'_{n+2} admissible. Thus

$$(2) \quad \sum_{i \in J} \|E_i a\|_{\tilde{T}} = \frac{1}{2} \sum_{i \in J} \sum_{p=1}^{k_i} \|E_p^i a\|_{\tilde{T}} \\ \leq \frac{1}{2} \sup \left\{ \sum_{i=1}^{\ell} \|F_i a\|_{\tilde{T}} : \{F_1, \dots, F_\ell\} \text{ } \mathcal{G}'_{n+2}\text{-admissible} \right\}.$$

From the inductive hypothesis and inequalities (1) and (2) we get

$$\|a\|_{\tilde{T}} \leq \sum_{i=0}^n \frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+1}} \left(\rho'_{n+1}(a) + \frac{1}{2} \sup \left\{ \sum_{i=1}^{\ell} \|F_i a\|_{\tilde{T}} \right\} \right) \\ = \sum_{i=0}^{n+1} \frac{\rho'_i(a)}{2^i} + \frac{1}{2^{n+2}} \sup \left\{ \sum_{i=1}^{\ell} \|F_i a\|_{\tilde{T}} \right\},$$

where both suprema are taken over all \mathcal{G}'_{n+2} -admissible sets $\{F_1, \dots, F_\ell\}$. This completes the induction. \square

Endow $[\mathbb{N}]^{<\infty}$ with the product topology inherited from $2^{\mathbb{N}}$. If \mathcal{F} is a closed subset of $[\mathbb{N}]^{<\infty}$, let \mathcal{F}' be the set of all limit points of \mathcal{F} . Define a transfinite sequence of sets $(\mathcal{F}^{(\alpha)})_{\alpha < \omega_1}$ as follows: $\mathcal{F}^{(0)} = \mathcal{F}$, $\mathcal{F}^{(\alpha+1)} = (\mathcal{F}^{(\alpha)})'$ for all $\alpha < \omega_1$; $\mathcal{F}^{(\alpha)} = \bigcap_{\beta < \alpha} \mathcal{F}^{(\beta)}$ if α is a countable limit ordinal.

Definition 7. ([9]) Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\infty}$ be regular. Define $\iota(\mathcal{F})$ to be the unique countable ordinal α such that $\mathcal{F}^{(\alpha)} = \{\emptyset\}$.

Let $F \in \mathcal{M}[\mathcal{N}]$, $F \neq \emptyset$, where \mathcal{M} and \mathcal{N} are regular families. There exists a largest $k \in F$ such that $F \cap [1, k] \in \mathcal{N}$. Set $F_1 = F \cap [1, k]$. If F_1, F_2, \dots, F_{n-1} have been defined and $F \setminus \bigcup_{i=1}^{n-1} F_i \neq \emptyset$, set $F_n = \left(F \setminus \bigcup_{i=1}^{n-1} F_i \right) \cap [1, k']$, where k' is the largest integer in F such that $\left(F \setminus \bigcup_{i=1}^{n-1} F_i \right) \cap [1, k'] \in \mathcal{N}$. Since F is finite, there exists an n such that $F = \bigcup_{i=1}^n F_i$. Now $F \in \mathcal{M}[\mathcal{N}]$ implies that there exists an \mathcal{M} -admissible collection $\{G_1, \dots, G_m\}$ such that $F = \bigcup_{j=1}^m G_j$ and $G_j \in \mathcal{N}$, $1 \leq j \leq m$. By the choice of F_i , and the fact that \mathcal{N} is hereditary, it is easy to see that $\min G_i \leq \min F_i$, $1 \leq i \leq n$. Thus $\{F_1, \dots, F_n\}$ is \mathcal{M} -admissible, as \mathcal{M} is spreading. We call $(F_i)_{i=1}^n$ the standard representation of F (as an element of $\mathcal{M}[\mathcal{N}]$).

Remark 8. Suppose that $(F_i)_{i=1}^n$ and $(G_i)_{i=1}^m$ are the standard representations of F and G respectively. If $\ell, k \in \mathcal{N}$ are such that $F \cap [1, \ell] = G \cap [1, \ell]$ and $\max F_k \leq \ell$, then by construction, $F_i = G_i$, $1 \leq i \leq k$.

Lemma 9. Let $\mathcal{M}, \mathcal{N} \subseteq [\mathbb{N}]^{<\infty}$ be regular. Suppose that $\iota(\mathcal{N}) = \alpha$, then $(\mathcal{M}[\mathcal{N}])^{(\alpha)} = (\mathcal{M}^{(1)})[\mathcal{N}]$.

Proof. Let $F \in (\mathcal{M}^{(1)})[\mathcal{N}]$, then F can be written as $F = \bigcup_{i=1}^n F_i$, where $F_1 < F_2 < \dots < F_n$, $F_1, \dots, F_n \in \mathcal{N}$, and $\{\min F_1, \dots, \min F_n\} \in \mathcal{M}^{(1)}$. In particular, there exists $k_0 > \max F$ such that $\{\min F_1, \dots, \min F_n, k\} \in \mathcal{M}$ for all $k \geq k_0$. Therefore,

$$(3) \quad \text{for all } G \in \mathcal{N}, \min G \geq k_0, F \cup G \in \mathcal{M}[\mathcal{N}].$$

Note that as \mathcal{N} is spreading,

$$(4) \quad \iota(\{G \in \mathcal{N} : \min G \geq k_0\}) = \iota(\mathcal{N}) = \alpha.$$

From (3),

$$(\{F \cup G : G \in \mathcal{N}, \min G \geq k_0\})^{(\alpha)} \subseteq (\mathcal{M}[\mathcal{N}])^{(\alpha)}.$$

But from (4), $F \in (\{F \cup G : G \in \mathcal{N}, \min G \geq k_0\})^{(\alpha)}$. Hence $F \in (\mathcal{M}[\mathcal{N}])^{(\alpha)}$.

Conversely, we prove by induction that $(\mathcal{M}[\mathcal{N}])^{(\gamma)} \subseteq ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\gamma)})$ for all $\gamma \leq \alpha$. The cases where $\gamma = 0$ is clear.

Suppose the claim is true for some $\gamma < \alpha$. Let $F \in (\mathcal{M}[\mathcal{N}])^{(\gamma+1)}$ with standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. Choose a sequence (G_k) in $(\mathcal{M}[\mathcal{N}])^{(\gamma)} \subseteq ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\gamma)})$ that converges nontrivially to F . We may assume that $G_k \cap [1, \min F_n] = F \cap [1, \min F_n]$ for all k . Now we may write $G_k = P_k \cup Q_k$, where $P_k < Q_k$, $P_k \in (\mathcal{M}^{(1)})[\mathcal{N}]$ and $Q_k \in \mathcal{N}^{(\gamma)}$. Let $P = \bigcup_{i=1}^{n-1} F_i$. Note that $P \in (\mathcal{M}^{(1)})[\mathcal{N}]$. We consider two cases.

Case 1. There exists k such that $\min F_n \in P_k$.

In this case, $P \cap [1, \max F_{n-1}] = F \cap [1, \max F_{n-1}] = G_k \cap [1, \max F_{n-1}] = P_k \cap [1, \max F_{n-1}]$. It is clear that $(F_i)_{i=1}^{n-1}$ is the standard representation of P as an element of $(\mathcal{M}^{(1)})[\mathcal{N}]$. By Remark 8, the standard representation of P_k as an element of $(\mathcal{M}^{(1)})[\mathcal{N}]$ has the form $(F_1, \dots, F_{n-1}, P_k^n, \dots, P_k^m)$. In particular,

$$\{\min F_1, \dots, \min F_{n-1}, \min F_n\} = \{\min F_1, \dots, \min F_{n-1}, \min P_k^n\} \in \mathcal{M}^{(1)}.$$

Thus $F = \bigcup_{i=1}^n F_i \in (\mathcal{M}^{(1)})[\mathcal{N}] \subseteq ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\gamma+1)})$, as required.

Case 2. Suppose $\min F_n \notin P_k$ for all $k \in \mathbb{N}$.

In this case, $G_k \cap [\min F_n, \infty) \subseteq Q_k$ for all k . Hence $G_k \cap [\min F_n, \infty) \in \mathcal{N}^{(\gamma)}$ for all k . Furthermore, $G_k \cap [\min F_n, \infty)$ converges to $F \cap [\min F_n, \infty) = F_n$ nontrivially. Thus $F_n \in \mathcal{N}^{(\gamma+1)}$. Therefore $F = P \cup F_n \in ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\gamma+1)})$, as required.

Suppose $\gamma \leq \alpha$ is a limit ordinal and the result holds for all $\eta < \gamma$. Let $F \in (\mathcal{M}[\mathcal{N}])^{(\gamma)}$ have standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. By the inductive hypothesis, for each $\eta < \gamma$, $F = P_\eta \cup Q_\eta$, where $P_\eta < Q_\eta$, $P_\eta \in (\mathcal{M}^{(1)})[\mathcal{N}]$ and $Q_\eta \in \mathcal{N}^{(\eta)}$. By the argument in case 1 above, if there exists η such that $\min F_n \in P_\eta$, then $F \in (\mathcal{M}^{(1)})[\mathcal{N}] \subseteq ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\eta)})$. Otherwise, $F_n \subseteq Q_\eta \in \mathcal{N}^{(\eta)}$ for all $\eta < \gamma$. Hence $F = \left(\bigcup_{i=1}^{n-1} F_i\right) \cup F_n \in ((\mathcal{M}^{(1)})[\mathcal{N}], \mathcal{N}^{(\gamma)})$. This completes the induction. \square

Proposition 10. Let $\mathcal{M}, \mathcal{N} \subseteq [\mathbb{N}]^{<\omega_1}$ be regular. Suppose that $\iota(\mathcal{N}) = \alpha$. Then for all $\beta < \omega_1$, $(\mathcal{M}[\mathcal{N}])^{(\alpha, \beta)} = (\mathcal{M}^{(\beta)})[\mathcal{N}]$.

Proof. The proof is by induction on β . The case $\beta = 0$ is clear. Suppose the result is true for some β . Then

$$\begin{aligned}
(\mathcal{M}[\mathcal{N}])^{(\alpha \cdot (\beta+1))} &= (\mathcal{M}[\mathcal{N}])^{(\alpha \cdot \beta + \alpha)} \\
&= \left((\mathcal{M}[\mathcal{N}])^{(\alpha \cdot \beta)} \right)^{(\alpha)} \\
&= \left((\mathcal{M}^{(\beta)})[\mathcal{N}] \right)^{(\alpha)} \text{ by the inductive hypothesis,} \\
&= \left((\mathcal{M}^{(\beta)})^{(1)}[\mathcal{N}] \right) \text{ by Lemma 9,} \\
&= (\mathcal{M}^{(\beta+1)})[\mathcal{N}].
\end{aligned}$$

Suppose the proposition is true for all $\beta < \beta_0$, where $\beta_0 < \omega_1$ is some limit ordinal. Clearly,

$$(\mathcal{M}^{(\beta_0)})[\mathcal{N}] \subseteq \bigcap_{\beta < \beta_0} (\mathcal{M}^{(\beta)})[\mathcal{N}] = \bigcap_{\beta < \beta_0} (\mathcal{M}[\mathcal{N}])^{(\alpha \cdot \beta)} = (\mathcal{M}[\mathcal{N}])^{(\alpha \cdot \beta_0)}.$$

On the other hand, let $F \in \bigcap_{\beta < \beta_0} (\mathcal{M}^{(\beta)})[\mathcal{N}]$ have standard representation $(F_i)_{i=1}^n$ as an element of $\mathcal{M}[\mathcal{N}]$. It is clear that $(F_i)_{i=1}^n$ is also the standard representation of F as an element of $(\mathcal{M}^{(\beta)})[\mathcal{N}]$ for any $\beta < \beta_0$. In particular, $\{\min F_i : 1 \leq i \leq n\} \in \mathcal{M}^{(\beta)}$ for all $\beta < \beta_0$. Hence $\{\min F_i : 1 \leq i \leq n\} \in \mathcal{M}^{(\beta_0)}$. It follows that $F \in (\mathcal{M}^{(\beta_0)})[\mathcal{N}]$. This completes the proof. \square

It is well known that $\iota(\mathcal{S}_\gamma) = \omega^\gamma$ for all $\gamma < \omega_1$ ([1, Proposition 4.10]). The indices of \mathcal{F}_n and \mathcal{F}'_n can be computed readily with the help of Proposition 10.

Corollary 11. $\iota(\mathcal{F}_n) = \omega^{\alpha+\beta \cdot n}$, $\iota(\mathcal{F}'_n) = \omega^{\alpha+\beta \cdot n} \cdot 2$.

Before proceeding further, let us recall the relevant terminology concerning trees. A *tree* on a set X is a subset T of $\bigcup_{n=1}^\infty X^n$ such that $(x_1, \dots, x_n) \in T$ whenever $n \in \mathbb{N}$ and $(x_1, \dots, x_{n+1}) \in T$. These are the only kind of trees we will consider. A tree T is *well-founded* if there is no infinite sequence (x_n) in X such that $(x_1, \dots, x_n) \in T$ for all n . Given a well-founded tree T , we define the *derived tree* $D(T)$ to be the set of all $(x_1, \dots, x_n) \in T$ such that $(x_1, \dots, x_n, x) \in T$ for some $x \in X$. Inductively, we let $D^0(T) = T$, $D^{\alpha+1}(T) = D(D^\alpha(T))$, and $D^\alpha(T) = \bigcap_{\beta < \alpha} D^\beta(T)$ if α is a limit ordinal. The *order* of a well-founded tree T is the smallest ordinal $o(T)$ such that $D^{o(T)}(T) = \emptyset$. If E is a Banach space and $1 \leq K < \infty$, an ℓ^1 - K tree on E is a tree T on $S(X) = \{x \in E : \|x\| = 1\}$ such that $\|\sum_{i=1}^n a_i x_i\| \geq K^{-1} \sum_{i=1}^n |a_i|$ whenever $(x_1, \dots, x_n) \in T$ and $(a_i) \subseteq \mathbb{R}$. If E has a basis (e_i) , a *block tree* on E is a tree T on E so that every $(x_1, \dots, x_n) \in T$ is a finite block basis of (e_i) . An ℓ^1 - K -block tree on E is a block tree that is also an ℓ^1 - K tree. The index $I(E, K)$ is defined to be $\sup\{o(T) : T \text{ is an } \ell^1\text{-}K \text{ tree on } E\}$. If E has a basis (e_i) , the index $I_b(E, K)$ is defined similarly, with the supremum taken over all ℓ^1 - K block trees. The Bourgain ℓ^1 -index of E is the ordinal $I(E) = \sup\{I(E, K) : 1 \leq K < \infty\}$. The index $I_b(E)$ is defined similarly. Bourgain proved that if E is a separable Banach space not containing a copy of ℓ^1 , then $I(E) < \omega_1$ [3]. More on these and related indices can be found in [7] and [2].

Proposition 12. *Let T be a well-founded block tree on some basis (e_i) . Define*

$$\mathcal{F}(T) = \{\{\max \text{supp } x_i : i = 1, \dots, n\} : (x_1, x_2, \dots, x_n) \in T\}.$$

and

$$\mathcal{G}(T) = \{G : \exists F \in \mathcal{F}(T), f : \mathbb{N} \rightarrow \mathbb{N} \text{ strictly increasing, such that } G \subseteq f(F)\}.$$

Then $\iota(\mathcal{G}(T)) \geq o(T)$.

Proof. Let $\xi = o(T)$. The proof is by induction on ξ . If $o(T) = 1$, then $\mathcal{G}(T) \supseteq \{\{k\} : k \geq n\}$ for some $n \in \mathbb{N}$. Therefore $(\mathcal{G}(T))^{(1)} \supseteq \{\emptyset\}$ and hence $\iota(\mathcal{G}(T)) \geq 1 = o(T)$.

Suppose the proposition holds for some $\xi < \omega_1$. Let T be a well founded block tree with $o(T) = \xi + 1$. For each $(x) \in T$, let

$$T_x = \bigcup_{n=1}^{\infty} \{(x_1, \dots, x_n) : (x, x_1, \dots, x_n) \in T\}.$$

According to [3, Proposition 4], $o(T) = \sup_{(x) \in T} \{o(T_x) + 1\}$. Therefore, there exists $(x_0) \in T$ such that $o(T_{x_0}) = \xi$. By the inductive hypothesis, $\iota(\mathcal{G}(T_{x_0})) \geq \xi$. Let $k_0 = \max \text{supp } x_0$. Then $G \in \mathcal{G}(T_{x_0})$ implies $\{k_0\} \cup G \in \mathcal{G}(T)$. Thus $\{k_0\} \in (\mathcal{G}(T))^{(\xi)}$. Since $(\mathcal{G}(T))^{(\xi)}$ is spreading, $\{k\} \in (\mathcal{G}(T))^{(\xi)}$ for all $k \geq k_0$. It follows that $\emptyset \in (\mathcal{G}(T))^{(\xi+1)}$. Hence $\iota(\mathcal{G}(T)) \geq \xi + 1 = o(T)$.

Suppose $o(T) = \xi_0$, where ξ_0 is a countable limit ordinal and the proposition holds for all $\xi < \xi_0$. Since $o(T) > \xi$ for all $\xi < \xi_0$. By the inductive hypothesis, $\iota(\mathcal{G}(T)) > \xi$ for all $\xi < \xi_0$. Hence $\iota(\mathcal{G}(T)) \geq \xi_0 = o(T)$. This completes the induction. \square

If $(x_k)_{k=1}^n$ and $(y_k)_{k=1}^n$ are sequences in possibly different normed spaces, and $0 < K < \infty$, we write $(x_k)_{k=1}^n \succeq_K (y_k)_{k=1}^n$ to mean $K \|\sum_{k=1}^n a_k x_k\| \geq \|\sum_{k=1}^n a_k y_k\|$ for all $(a_k) \subseteq \mathbb{R}$.

Theorem 13. $I(\tilde{T}_{\alpha, \beta}) = I_b(\tilde{T}_{\alpha, \beta}) = \omega^{\alpha + \beta \cdot \omega}$.

Proof. If $I_b(\tilde{T}_{\alpha, \beta}) > \omega^{\alpha + \beta \cdot \omega}$, then $I_b(\tilde{T}_{\alpha, \beta}, K) > \omega^{\alpha + \beta \cdot \omega}$ for some $K > 1$. Hence there exists an ℓ^1 - K -block tree T on $\tilde{T}_{\alpha, \beta}$ such that $o(T) = \xi > \omega^{\alpha + \beta \cdot \omega}$. Given $F \in \mathcal{F}(T)$, there exists $(x_1, x_2, \dots, x_n) \in T$ such that $F = \{\max \text{supp } x_i\}_{i=1}^n$. According to Proposition 4, $(e_k)_{k \in F} \succeq^2 (x_1, x_2, \dots, x_n)$, where $(e_k)_{k=1}^\infty$ is the unit vector basis of $\tilde{T}_{\alpha, \beta}$. Since $(x_1, x_2, \dots, x_n) \in T$, $(x_1, x_2, \dots, x_n) \succeq^K \ell^1(|F|)$ -basis. Therefore, $(e_k)_{k \in F} \succeq^{2K} \ell^1(|F|)$ -basis for all $F \in \mathcal{F}(T)$. Since it is clear that $\|\sum a_k e_{f(k)}\|_{\tilde{T}} \geq \|\sum a_k e_k\|_{\tilde{T}}$ for all $(a_k) \in c_{00}$ whenever $f : \mathbb{N} \rightarrow \mathbb{N}$ is strictly increasing, it follows that $(e_k)_{k \in G} \succeq^{2K} \ell^1(|G|)$ -basis for all $G \in \mathcal{G}(T)$. By Proposition 12, $(\mathcal{G}(T))^{(\omega^{\alpha + \beta \cdot \omega} + 1)} \neq \emptyset$. Thus by [6, Corollary 1.2], there exists $L \in [\mathbb{N}]$ such that $\mathcal{S}_{\alpha + \beta \cdot \omega} \cap [L]^{<\infty} \subseteq \mathcal{G}(T)$. Hence, for all $(a_k) \in c_{00}$ and all $G \in \mathcal{S}_{\alpha + \beta \cdot \omega} \cap [L]^{<\infty}$,

$$(5) \quad \left\| \sum_{k \in G} a_k e_k \right\|_{\tilde{T}} \geq \frac{1}{2K} \sum_{k \in G} |a_k|.$$

Choose $m \in \mathbb{N}$ such that $2^m > 2K$. According to Corollary 11, $\iota(\mathcal{F}_i^L) = \omega^{\alpha + \beta \cdot i} \cdot 2$ for all $i = 1, 2, \dots, m$. Applying [6, Corollary 1.2], we obtain $M \in [L]$ such that $\mathcal{F}_i^L \cap [M]^{<\infty} \subseteq \mathcal{S}_{\alpha + \beta \cdot m + 1}$ for all $i = 1, 2, \dots, m$. By [9, Proposition 3.6], there exists $F \in \mathcal{S}_{\alpha + \beta \cdot \omega}(M) \subseteq \mathcal{S}_{\alpha + \beta \cdot \omega} \cap [M]^{<\infty}$ and $(a_j)_{j \in F} \subseteq \mathbb{R}^+$ such that $\sum a_j = 1$ and if

$G \subseteq F$ with $G \in \mathcal{S}_{\alpha+\beta \cdot m+1}$, then $\sum_{j \in G} a_j < \frac{1}{8K}$. Consider $x = \sum_{j \in F} a_j e_j \in \tilde{T}_{\alpha, \beta}$. If $1 \leq i \leq m$ and $I \in \mathcal{F}'_i$, then $I \cap F \in \mathcal{F}'_i \cap [M]^{<\infty} \subseteq \mathcal{S}_{\alpha+\beta \cdot m+1}$. Hence $\sigma_I(x) = \sigma_{I \cap F}(x) < \frac{1}{8K}$. It follows that $\rho'_i(x) \leq \frac{1}{8K}$ for $1 \leq i \leq m$. By Proposition 6,

$$\begin{aligned} \|x\|_{\tilde{T}} &\leq \sum_{i=0}^m \frac{\rho'_i(x)}{2^i} + \frac{1}{2^{m+1}} \sup \left\{ \sum_{i=1}^j \|E_i x\|_{\tilde{T}} : \{E_1, \dots, E_j\} \text{ } \mathcal{G}'_{m+1}\text{-admissible} \right\} \\ &\leq \sum_{i=0}^m \frac{\frac{1}{8K}}{2^i} + \frac{1}{2^{m+1}} \|x\|_{\ell^1} < \frac{1}{2K}, \end{aligned}$$

contrary to (5). This proves that $I_b(\tilde{T}_{\alpha, \beta}) \leq \omega^{\alpha+\beta \cdot \omega}$. On the other hand, according to Proposition 5, for any $n \in \mathbb{N}$, $\|a\|_{\tilde{T}} \geq \frac{1}{2^n} \|a\|_{\mathcal{F}_n}$ for any $a \in c_{00}$. By Corollary 11, $\iota(\mathcal{F}_n) = \omega^{\alpha+\beta \cdot n}$. Therefore, there exists an ℓ^1 - 2^n -block basis tree T_n on $\tilde{T}_{\alpha, \beta}$ with $o(T_n) = \omega^{\alpha+\beta \cdot n}$. Hence $I_b(\tilde{T}_{\alpha, \beta}, 2^n) \geq \omega^{\alpha+\beta \cdot n}$. Thus $I_b(\tilde{T}_{\alpha, \beta}) = \sup_K I_b(\tilde{T}_{\alpha, \beta}, K) \geq \omega^{\alpha+\beta \cdot \omega}$. We conclude that $I_b(\tilde{T}_{\alpha, \beta}) = \omega^{\alpha+\beta \cdot \omega}$. As $I(\tilde{T}_{\alpha, \beta}) \geq I_b(\tilde{T}_{\alpha, \beta}) \geq \omega^\omega$, it follows from [7, Corollary 5.13] that $I(\tilde{T}_{\alpha, \beta}) = I_b(\tilde{T}_{\alpha, \beta})$. \square

For the final corollary, recall that the Schreier space X_α , $\alpha < \omega_1$, is the completion of c_{00} with respect to the norm $\|\cdot\|_{\mathcal{S}_\alpha}$.

Corollary 14. *Suppose α is a countable ordinal whose Cantor normal form is $\omega^{\alpha_1} \cdot m_1 + \dots + \omega^{\alpha_k} \cdot m_k$. If α_k is not a limit ordinal, then there exists a Banach space X such that $I(X) = \omega^\alpha$.*

Proof. If $\alpha_k = 0$, then α is a successor ordinal and $\iota(X_{\alpha-1}) = \omega^\alpha$ ([2]). If α_k is a successor ordinal, let $\gamma = \omega^{\alpha_1} \cdot m_1 + \dots + \omega^{\alpha_k} \cdot (m_k - 1)$ and $\eta = \omega^{\alpha_k - 1}$. By Theorem 13, $I(\tilde{T}_{\gamma, \eta}) = \omega^{\gamma+\eta \cdot \omega} = \omega^\alpha$. \square

Remark 15. The following analog of Proposition 4 for the space X_α , $\alpha < \omega_1$, holds obviously: If $(x_i)_{i=1}^p$ is a normalized block basis of the unit vector basis (e_k) of X_α , and $k_i = \max \text{supp } x_i$, $1 \leq i \leq p$, then $\|\sum_{i=1}^p a_i x_i\| \leq \|\sum_{i=1}^p a_i e_{k_i}\|$ for all $(a_i) \in c_{00}$. Therefore the arguments in Proposition 12 and Theorem 13 can be used to compute $I_b(X_\alpha)$ (with respect to the basis (e_k)).

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE 117543
E-mail address: `matlhh@nus.edu.sg`

MATHEMATICS AND MATHEMATICS EDUCATION, NATIONAL INSTITUTE OF EDUCATION, NANYANG
TECHNOLOGICAL UNIVERSITY, 1 NANYANG WALK, SINGAPORE 637616
E-mail address: `wktang@nie.edu.sg`